$$r^2s : J=24b\beta^3D+72b(\beta D-d)+24d(\delta+2b)-24b(\beta+D)^2,$$
$$s^2t : K=2d(\delta+2b)^2+[24b(\delta+2b)-16\beta d](\beta D-d)-24\beta^2(d\delta-d\beta)$$
$$-(24b^2D+8d^2)(\beta+D),$$
$$rs^2 : L=-72b^2D-24d^2-16\beta d(\beta+D)-8\beta^2(\beta D-d)-2\beta(\delta+2b)^2$$
$$+24b(\beta+D)(\delta+2b),$$
$$r^2t : M=[24d+8\beta(\beta+D)](\beta D-d)-72\beta(b\delta-d\beta)-24bD(\delta+2b)$$
$$+2(\beta+d)(\delta+2b)^2-8d(\beta+D)^2,$$

the coefficients of rt^2, st^2 and rst not being exhibited.

We require that the coefficients A, B, ... of the Hessian shall be proportional to the corresponding coefficients of $\triangle$. It will not be necessary to examine all of these conditions. We exclude at present the special case in which the reference field F has a modulus* 2 or 3, so that we may divide by these numbers. Since there is no term r^2s in $\triangle$, we have $J=0$,

$$2b\beta D-b\beta^2-bD^2-bd+d\delta=0. \tag{7}$$

The condition $A : B=1 : b$ is then satisfied. From $C : A=(b\delta-d\beta) : 1$,

$$\beta^4d-\beta^3b\delta-\beta^3dD+\beta^2b\delta D-\beta Dd^2+\beta b^2D^2+2\delta dDb-2b^2dD-b^2D^3-d^3=0. \tag{8}$$

From $K : A=d : 1$,

$$[d\beta+3b(\delta+2b)](\beta D-d)+d(\delta+2b)^2-\beta d(\beta+D)^2-3\beta^2(b\delta-d\beta)$$
$$-(d^2+3b^2D)(\beta+D)=0. \tag{9}$$

Now $b\neq0$ in view of the assumed irreducibility of

$$x^3-\beta x-b. \tag{10}$$

We assume here that $\beta\neq0$. We may set

$$\delta=xb,\ D=yb,\ d=z\beta^2. \tag{11}$$

Then (7) has the factor $b\beta^2$, and we get

$$xz-z=(y-1)^2. \tag{12}$$

Next, we multiply (8) and (9) by d, and eliminate $d\delta$ by means of (7). In the resulting conditions make the substitution (11). After suitable factorizations, we get

$$\beta^5(z+1)G=0,\ \beta^4(y+2)G=0, \tag{13}$$

where $$G=b^2(y-1)^3-\beta^3z^2(z+y-1).$$

*The reader may hold F to be an algebraic field; it is only for completeness that modular fields are included (with proper restrictions).

If $G=0$, set $Y=b(y-1)$, $Z=\beta z$. Then $bG=0$ becomes

$$Y^3-\beta YZ^2-bZ^3=0.$$

But (10) is irreducible. Hence $Y=Z=0$, so that $y=1$, $z=0$. Thus $D=\beta$, $d=0$. Now $L : A=-\beta : 1$. Thus

$$0=L+\beta A=-8\beta(\delta-b)^2.$$

Hence $\delta=b$. Hence relations (3) reduce to (2), so that *the algebra is a field.*

Let next $G\neq 0$. Then $z=-1$, $y=-2$, $x=-8$, by (13) and (12). Thus $\delta=-8b$, $D=-2\beta$, $d=-\beta^2$ by (11). The remaining conditions are seen to be satisfied. Indeed, the Hessian now equals

$$(32\beta^3-216b^2)\triangle.$$

The multiplication table of the resulting algebra is given by (1) and

$$i^2=j,\ ij=ji=b+\beta i,\ j^2=-\beta^2-8bi-2\beta j. \tag{14}$$

A direct proof that division is always uniquely possible in this remarkable algebra is given in the papers cited above.

Consider next the case $\beta=0$. Eliminating $d\delta$ between (7) and (8), we get $b^2D^3-d^3=0$. If $d\neq 0$, set $x=bD/d$. Then $x^3-b=0$, contrary to the irreducibility of (10). Hence $d=0$, $D=0$. Set $\lambda=-(\delta+2b)$. Then

$$\triangle=r^3+rst\lambda+s^3b+t^3b\delta,\quad H=-6r^3\lambda^2-6t^3\lambda^2b\delta-6s^3\lambda^2b+rst(2\lambda^3+6^3b^2\delta).$$

Hence H is a constant times $\triangle$ if and only if

$$2\lambda^3+6^3b^2\delta=-6\lambda^3,$$

viz., if $(\delta-b)^2(\delta+8b)=0$. The resulting algebras are again (2) and (14).

For completeness, we treat the excluded case* of a field with modulus 3. Let first $\beta\neq 0$. From $B : A=b : 1$,

$$b(\beta-D)^2=d(\delta+2b). \tag{15}$$

Reducing (9) modulo 3 and eliminating $\delta+2b$ by (15), we get

$$(\beta-D)[d^2-\beta d^2(\beta-D)+b^2(\beta-D)^3]=0.$$

If the second factor vanished, (10) would have a rational root

$$x=-b(\beta-D)/d.$$

*For modulus 2, $H\equiv 0$. A discussion shows that the algebra is a field.

Hence $\beta=D$. Hence $A=-\beta^3$, $L=\beta^4+\beta(\delta+2b)^2$. Hence $L:A=-\beta:1$ gives $\delta\equiv b \pmod 3$. Then (8) becomes $d^2(d+\beta^2)\equiv 0$. Hence $d=0$ or $-\beta^2$, giving again algebras (2) and (14). Finally, let $\beta=0$. Then $A\equiv 0$, so that the Hessian must be identically zero. From C, K, we get

$$bdD(\delta+2b)+b^2D^3+d^3=0,\quad d^2D=d(\delta+2b)^2.$$

If $d\neq 0$, we eliminate $\delta+2b$ and get $(b^2D^3-d^3)^2=0$. Hence $(bDd^{-1})^3-b=0$, contrary to the irreducibility of (10). Hence $d=0$, $D=0$. Then $H=(\delta+2b)^3rst$, so that $\delta\equiv b$. The algebra is again the field algebra.

In summary, *the only algebras with the required properties are the obvious field-algebra and the remarkable non-field algebra* (14).

The University of Chicago, *September, 1906.*

THE DIVISION OF ANGLES INTO n EQUAL PARTS.

By J. SAMSONOFF, New York City.

Principle: If by the method given later one has constructed the geometrical loci of vertices of triangles with base XA, whose remote angle at the base is equal respectively to one, two, three, ..., $(n-1)$ times the angle at the vertex of the respective triangle, any angle CAY is geometrically divisible by 2, 3, ..., n.

Given: Base XA and curves a, b, c, d, ..., the geometrical loci of vertices of triangles, where the remote

$$\angle aXA=\angle XaA,\ \angle bXA=2\angle XbA,\ \ldots,$$
$$\angle eXA=(n-1)\angle XeA.$$

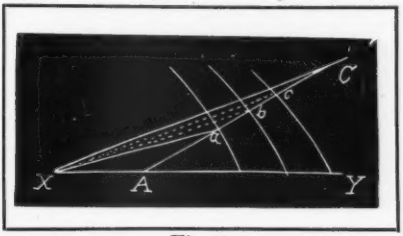


Fig. 1.

To prove: $\angle CAY$ is geometrically divisible into 2, 3, 4, ..., n equal parts.

Proof: (1) In $\triangle XaA$, $\angle aXA=\angle XaA$ (by hypothesis), $\angle CAY=\angle aXA+\angle XaA$ (the exterior angle of $\triangle aXA$). Therefore the measure of $\angle XaA$ must be equal to $\frac{1}{2}$ of the measure of $\angle CAY$.

(2) In $\triangle XbA$, $\angle bXA=2\angle XbA$ (by hypothesis), $\angle CAY=\angle bXA+\angle XbA$ (the exterior angle of $\triangle XbA$). Therefore the measure of $\angle XbA$ must be equal to $\frac{1}{3}$ of the measure of $\angle CAY$.

(n) In $\triangle XeA$, $\angle eXA=(n-1)\angle XeA$ (by hypothesis), $\angle CAY=\angle eXA+\angle XeA$ (the exterior angle of $\triangle eXA$). Therefore the measure of $\angle XeA$ must be equal to $1/n$ of the measure of $\angle CAY$.

Hence the problem to divide an angle into any number of equal parts depends upon the solution of the problem of drawing the respective curve.

Analysis of the method of drawing a curve as the geometrical locus of vertices of triangles where the remote angle at the base is equal to $(n-1)$ times the angle at the vertex.

Let XY be a given straight line from which (in relation to the points X and A) a n-sectorial curve (a) is drawn. In Fig. 2, $n=6$. Connect any point a_6 of the curve with the relative points X and A.

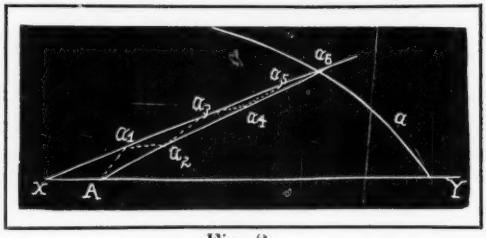


Fig. 2.

In $\triangle a_6XA$, $\angle a_6XA = 5(\angle Xa_6A)$, by hypothesis. Take point A as a center and with AX as a radius circumscribe a circle which intersects line a_6X in a_1. In $\triangle Aa_1a_6$, the exterior $\angle Xa_1A$ is equal to $\angle a_1Aa_6 + \angle Aa_6a_1$; but $\angle Xa_1A = \angle a_1XA$ by construction, and $\angle a_6XA = 5(\angle Xa_6A)$ by hypothesis. Hence $5(\angle Xa_6A) = \angle a_1Aa_6 + \angle Xa_6A$, or $\angle a_1Aa_6 = 4(\angle Xa_6A)$. For general n, $\sphericalangle a_nXA = (n-1)(\sphericalangle Xa_nA)$, $\sphericalangle a_1Aa_n = (n-2)(\sphericalangle Xa_nA)$.

Take point a_1 as a center and with $a_1A = AX$ as a radius circumscribe a circle which will intersect line AC at a_2; then, reasoning as above, we will find that $\angle a_na_1a_2 = (n-3)(\angle Xa_nA)$. Repeating this construction, we will form new triangles, where one angle at the base will equal respectively $(n-4)$, $(n-5)$, ..., 4, 3, 2, and 1 times $\angle Xa_nA$. The last triangle will also be an isosceles triangle. Therefore, we must consider $\triangle Xa_6A$ as a sum of a series of isosceles triangles, where all sides are equal to XA.

The nearer to XY point a_6 on the curve is taken, the less is the difference between XY and Xa_6, the less the difference between AY and Aa_6; and when point a_6 coincides with Y, the broken line $Aa_1a_2 \ldots a_6$ straightens itself and coincides with AY. Hence the relative points A and X for a required curve are: For a bisectorial curve, $XA = AY$; for a trisectorial curve, $XA = \frac{AY}{2}$; ...; for a n-sectorial curve, $XA = \frac{AY}{n-1}$.

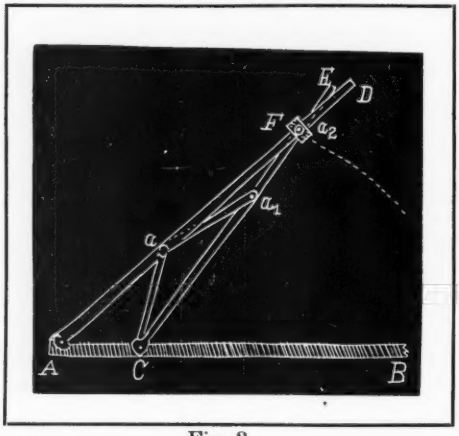


Fig. 3.

It remains to construct the instrument for drawing the curves. This instrument (Fig. 3) consists of a ruler AB supplied with pins so that when ruler AB is placed in coin-

cidence with any straight line it will become fixed. At the points A and C the rulers AD and CE are attached in hinge-like fashion so that they may be freely movable. These rulers are also supplied with slits along their whole length. Also, at the point C hinges a folded ruler $Caa_1a_2a_3\ldots$ consisting of parts Ca, aa_1, ..., each part being equal to AC. The rulers AD, CE, and the folded ruler $Caa_1a_2\ldots$ are held together by a common handle F, which may be removed and replaced. This handle is supplied with a pencil for drawing.

When the ruler AB is placed on a straight line and the instrument is folded, the straight line is visible through the open slits of rulers CE and AD.

With this instrument we are able to solve the problem:

Draw a curve as the geometrical locus of vertices of triangles, the remote angle at the common base of which is $(n-1)$ times the angle at the vertex.

Solution: Apply instrument AB (Fig. 4) to the line XY. Bring down the ruler CE until its slit coincides with line XY. Unfold the folded ruler $Caa_1a_2a_3\ldots$ and take $(n-1)$ parts of it. Connect the odd points of the folded ruler with the ruler CE. Bring down the ruler AD until its slit coincides with line XY, and connect the even points of the folded ruler with the ruler AD. Now apply the handle F at the $(n-1)$st part of the folded ruler, fixing the rulers CE and AD. Move handle F (supplied with a pencil). The instrument will draw the required curve.

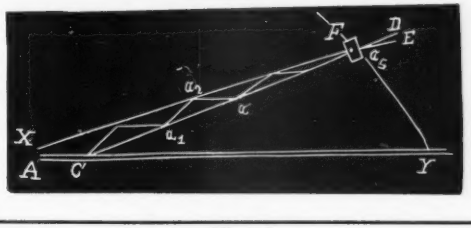


Fig. 4.

Let us prove that the drawn curve is the one required.

We have to prove that $\angle FCY$ is divisible geometrically into n equal parts.

Proof: $\triangle XFC$ is the sum of a series of $(n-1)$ isosceles triangles where the angles at the bases are successively equal to 1, 2, 3, 4, ..., $(n-1)$ times the angle at the vertex of $\triangle XFC$, by the construction of the instrument. [The points a, a_2, a_2, ..., of the folded ruler $Caa_1a_2a_3\ldots$ will slide between the directrices AD and CE.] Therefore, in $\triangle XFC$, $\angle FXC=(n-1)(\angle XFC)$. Hence $\angle FCY$ is divisible into n equal parts.

But the triangles formed by moving handle F of the two connected directrices will include the series of $(n-1)$ isosceles triangles. Therefore the formed curve is the required one.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

269. Proposed by C. N. SCHMALL, College of the City of New York.

Two ferry-boats started simultaneously from opposite sides of a river and one being faster than the other, they met 720 yards from the shore. Each boat remained 10 minutes in its slip to change passengers and started on its return trip, when it was found that they met again 400 yards from the other shore. What is the width of the river?

Solution by THEODORE L. De LAND, Treasury Department, Washington, D. C.

Let x=the width of the river in yards; y=the speed of the slower boat in yards per minute, and z=the speed of the faster boat in yards per minute.

Then $720 \div y$=the time for the slower boat to travel 720 yards, and $(x-720) \div z$=the time for the faster boat to travel $x-720$ yards; and as the two boats now meet, we equate the time, and have after reduction:

$$720z=(x-720)y \ldots (1).$$

Then $(x-720) \div y+10+400 \div y$=the time for the slower boat to reach the shore, wait ten minutes for passengers, and travel 400 yards; and $720 \div z+10+(x-400) \div z$=the time for the faster boat to reach the shore, wait ten minutes for passengers, and travel $x-400$ yards; and as the two boats now meet again, we equate the time as before, and have after reduction:

$$(x+320)y=(x-320)z \ldots (2).$$

Multiply (1) by (2), member by member, to eliminate y and z, and we have:

$$720(x+320)=(x-720)(x-320) \ldots (3).$$

From (3) we find $x=0$ or 1760 yards=1 mile, the width of the river.

From (1), after substituting for x, we have, $y : z :: 9 : 13$; or the speed of the slower boat is to the speed of the faster boat as 9 is to 13.

Also solved by L. E. Newcomb, Daniel B. Northrop, A. H. Holmes, J. E. Sanders, G. W. Greenwood, Frank M. Dyzer, and L. H. Rice.

270. Proposed by GEORGE H. HALLETT, Ph. D., Assistant Professor of Mathematics in The University of Pennsylvania, Philadelphia, Pa.

Find the simplest integral form of the sum $y(y-1) \ldots (y-x)+2y(2y-1) \ldots (2y-x)+ \ldots +zy(zy-1) \ldots (zy-x)$.

No solution has been received.

271. Proposed by L. E. NEWCOMB, Los Gatos, California.

Sum the series $\frac{a}{b}+\frac{a^3}{3b^3}+\frac{a^5}{5b^5}+\ldots$ to ∞, $b>a$.

I. Solution by A. H. HOLMES, Brunswick, Maine.

Let $S=\frac{a}{b}+\frac{a^3}{3b^3}+\frac{a^5}{5b^5}+$ etc., to infinity, $b>a$.

Put $\frac{a}{b}=c$, then $S=c+\frac{c^3}{3}+\frac{c^5}{5}+$ etc. $\quad\therefore\frac{dS}{dc}=1+c^2+c^4+$ etc.

Let $c^2=e$. Then $\frac{2cdS}{de}=1+e+e^2+e^3+$ etc. $=\frac{1}{1-e}$, $dS=\frac{de}{2c(1-e)}=\frac{dc}{1-c^2}$.

$\therefore S=\frac{1}{2}\log\left(\frac{1+c}{1-c}\right)=\frac{1}{2}\log\left(\frac{b+a}{b-a}\right)$.

II. Solution by J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md.

Subtracting $\log(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\ldots$ from $\log(1+x)=+x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\ldots$, we get $x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots=\frac{1}{2}\log\frac{1+x}{1-x}$.

Substituting $x=\frac{a}{b}$, we obtain $\frac{a}{b}+\frac{a^3}{3b^3}+\frac{a^5}{5b^5}+\ldots=\frac{1}{2}\log\frac{b+a}{b-a}$.

GEOMETRY.

294. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Apply the locus of $(x^2+y^2)^3=mx^3$ to the problem of finding a cube m times a given cube.

I. Solution by the PROPOSER.

Construct by points the locus of the equation. Take a value of x equal to the side of the given cube. Join the corresponding point of the curve to the origin. Denote this line by a. Construct the line numerically equal to the square root of a. Then evidently the cube of this line is mx^3; or this line is the side of a cube m times as large as the given cube.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

On $AB=\sqrt[3]{m}$ describe a circle. On AB lay off $AQ=x$, the edge of the given cube. At Q, erect a perpendicular intersecting the circle at P. Let $PQ=y$. Then since $AP^2=\sqrt[3]{(m)}AQ$, we have $x^2+y^2=\sqrt[3]{(m)}x$, and, therefore. $(x^2+y^2)^3=mx^3$. Hence, the cube on AP is always m times the cube on AQ.

296 (Incorrectly numbered 294). Proposed by JOHN JAMES QUINN, Ph. D., Scottdale, Pa.

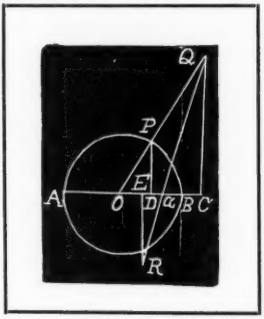
a) Suppose an indefinite line be pivoted at the end of a revolving radius whose center is the origin; and the initial position of the radius is coincident with the X-axis and the pivoted line perpendicular to it. As the radius revolves through equal amounts of arc the line moves to the right over corresponding equal intercepts on the X-axis. What is the equation of the locus of a point on the line whose distance from the end of the radius is equal to a diameter?

b) Show how the locus can be applied to the multisection of an angle.

c) Suppose the diameter be laid off in both directions.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let AB be a diameter of the given circle; $\angle POB=\theta=$angle OP makes with AB; a, the intersection of the line with AB; $PQ=PR=AB=2r$; $Ba=r\theta/90$; $Oa=r(1-\frac{1}{90}\theta)$; $Pa=r\sqrt{[1+(1-\frac{1}{90}\theta)^2-2(1-\frac{1}{90}\theta)\cos\theta]}=A$; $aQ=2r+A$; $aR=2r-A$; $PD=r\sin\theta$; $OD=r\cos\theta$.



$$QC=\frac{DP.Qa}{Pa}=\frac{r\sin\theta(2r+A)}{A}=y, \text{ for } Q \text{ locus.}$$

$$ER=\frac{r\sin\theta(2r-A)}{A}=y, \text{ for } R \text{ locus.}$$

$$aC=\frac{r}{A}(2r+A)(\cos\theta-1+\frac{\theta}{90}).$$

$$aE=\frac{r}{A}(2r-A)(\cos\theta-1+\frac{\theta}{90}).$$

$$OC=\frac{r(2r+A)\cos\theta-2r^2(1-\frac{1}{90}\theta)}{A}=x, \text{for } Q \text{ locus.}$$

$$OE=\frac{2r^2(1-\frac{1}{90}\theta)-r(2r-A)\cos\theta}{A}=x, \text{ for } R \text{ locus.}$$

$$\therefore \rho^2=x^2+y^2=5r^2\pm\frac{4r^3-4r^3\cos\theta(1-\frac{1}{90}\theta)}{A}.$$

The plus sign is for Q, the minus for R. The same curve will be generated in the opposite way for the diameter (part) to the left of O.

Let ϕ be the angle to be multisected. Then $r\phi/90$ is the part of BO cut off by the pivoted line. Let $\theta=\phi/m$; this value of θ gives Pa. Let $Ba=r\phi/90m$. With a as center and Pa as radius, intersect the circle at P. Draw OP. Then POB is the required angle, ϕ/m. So on for any section or any number of sections.

299. Proposed by G. W. GREENWOOD, M. A., Dunbar, Pa.

Show that the circle on any focal radius of an ellipse touches the auxiliary circle.

Solution by B. F. FINKEL, Ph. D., Drury College, Springfield, Mo.

Taking the left hand focus of ellipse as origin of rectangular coördinates, the axis of abscissas coinciding with the major axis of the ellipse, we have for the equation of the ellipse, auxiliar circle, and radius vector, respectively

$$\frac{(x-ae)^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1\ldots(1);\quad (x-ae)^2+y^2=a^2\ldots(2);\quad y=mx\ldots(3).$$

Solving (1) and (3), we find $x=\frac{a(1-e^2)[e\pm\sqrt{(1+m^2)}]}{1-e^2+m^2}=k$, say. Then $y=mk$. Hence, the coördinates of the center of the circle described on the focal radius are $\frac{1}{2}k$, $\frac{1}{2}mk$, and the equation of this circle is

$$(x-\tfrac{1}{2}k)^2+(y-\tfrac{1}{2}mk)^2=\tfrac{1}{4}k^2(1+m^2)\ldots(4).$$

Solving (2) and (4), we find that the discriminant of the resulting quadratic equations vanish, showing that the roots are equal. Hence, (4) intersects (2) in two coincident points and is, therefore, tangent to it.

CALCULUS.

222 (Incorrectly numbered 221, p. 117). Proposed by Professor F. ANDEREGG, Oberlin College, Oberlin, O.

If a, b, c, ... represent all the prime numbers 2, 3, 5, ... prove that

$$\left(1+\frac{1}{a^2}\right)\left(1+\frac{1}{b^2}\right)\left(1+\frac{1}{c^2}\right)\ldots\ldots=\frac{15}{\pi^2}.$$

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

$$\left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{5^2}\right)\left(1+\frac{1}{7^2}\right)\ldots\ \Big)$$

$$=\frac{\left(1-\frac{1}{2^4}\right)\left(1-\frac{1}{3^4}\right)\left(1-\frac{1}{5^4}\right)\left(1-\frac{1}{7^4}\right)\ldots}{\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\left(1-\frac{1}{7^2}\right)\ldots}=\frac{\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\ldots}{\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\ldots}$$

$$=\frac{\pi^5/6}{\pi^4/90}=\frac{15}{\pi^2}.$$

(See Vol. V, No. 5, May, 1898, page 134, third line from bottom, together with what precedes, for the complete solution. Also see Vol. XIII, No. 2, February, 1906, pages 40—41.)

223. Proposed by O. E. GLENN, Ph. D., Philadelphia, Pa.

Prove that $\underset{n=\infty}{\text{Lim.}}\ \frac{\sum_{\lambda=1}^{n}\lambda^k}{n^{k+1}}=\frac{1}{k+1}$.

I. Solution by G. W. GREENWOOD, M. A., Dunbar, Pa.

Putting $h=\frac{1}{n}$, we get $\underset{\substack{h=0\\ nh=0}}{\text{Lim.}}\ h[h^k+(2h)^k+(3h)^k+\ldots+(nh)^k]$

$$=\int_0^1 x^k dx=\frac{1}{k+1}.$$

II. Solution by EDWIN L. RICH, Schenectady, N. Y.

Here $\sum_{\lambda=1}^{n}\lambda^k=\frac{n^{k+1}-1}{k+1}$, treating the summation as an integration. Then we have $\underset{n=\infty}{\text{Limit}}\left(\frac{1}{(k+1)}\right)\left(\frac{n^{k+1}-1}{n^{k+1}}\right)$. Since this is in the form $\frac{\infty}{\infty}$, differentiate the numerator and denominator $k+1$ times, dividing numerator and denominator by $(k+1)$, (k), $(k-1)$, ..., $(k-k+1)$ successively, and we have

$$\underset{n=\infty}{\text{Lim.}}\ \frac{\sum_{\lambda=1}^{n}\lambda^k}{n^{k+1}}=\frac{1}{k+1}.$$

We might arrive at the same conclusion, by considering n finite, then $\frac{1}{k+1}\left(\frac{n^{k+1}-1}{n^{k+1}}\right)=\frac{1}{k+1}\left(1-\frac{1}{n^{k+1}}\right)$, which when $n=\infty$, approaches $\frac{1}{k+1}$ as a limit.

III. Solution by J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md.

One of the most general formulae for the summation of series is that of Euler, who has stamped his genius upon so many branches of mathematics. It is

$$\Sigma u=\int u\partial x+\tfrac{1}{2}u+\frac{B_1}{1.2}\cdot\frac{\partial u}{\partial x}-\frac{B_3}{4!}\frac{\partial^3 u}{\partial x^3}+\frac{B_5}{6!}\cdot\frac{\partial^5 u}{\partial x^5}-\ldots+\ldots$$

where B_1, B_3, B_5, denote the famous Bernoulli's Numbers $\frac{1}{6}$, $\frac{1}{30}$, $\frac{1}{42}$, etc. Putting $u=x^k$, we have

$$\Sigma x^k=\frac{x^{k+1}}{k+1}+\tfrac{1}{2}x^k+\frac{B_1}{2}\cdot\binom{n}{1}x^{k-1}-\ldots$$

Consequently we have

$$\frac{\Sigma n^k}{n^{k+1}}=\frac{n^{k+1}}{n^{k+1}}\left(\frac{1}{k+1}+\tfrac{1}{2}\cdot\frac{1}{n}+\tfrac{1}{6}\cdot\tfrac{1}{2}\cdot\frac{1}{n^2}+\ldots\right)=\frac{1}{k+1}+\tfrac{1}{2}\cdot\frac{1}{n}+\ldots$$

For $n=\infty$, all the terms save the first cancel, and we have $\frac{1}{k+1}$ as the limiting value. The sum of the kth powers of the natural series of numbers can also be found without the aid of calculus by the theory of undetermined coefficients.

227. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

Prove that $\int_0^\infty \tan^{-1}(\tan\alpha \sin x)\frac{dx}{x} = \frac{1}{2}\pi \log(\tan\alpha + \sec\alpha)$.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $\tan\alpha = m$. $\int_0^\infty \tan^{-1}(\tan\alpha \sin x)\frac{dx}{x} = \int_0^\infty \tan^{-1}(m \sin x)\frac{dx}{x}$.

$$\tan^{-1}(m \sin x) = m \sin x - \tfrac{1}{3}m^3\sin^3 x + \tfrac{1}{5}m^5\sin^5 x - \tfrac{1}{7}m^7\sin^7 x + \ldots$$

$$\sin^3 x = -\tfrac{1}{4}(\sin 3x - 3\sin x)$$

$$\sin^5 x = \tfrac{1}{16}(\sin 5x - 5\sin 3x + 10\sin x)$$

$$\sin^7 x = \tfrac{1}{64}(\sin 7x - 7\sin 5x + 21\sin 3x - 35\sin x)$$

. . . . . .

$$\int_0^\infty \frac{\sin x\, dx}{x} = \frac{\pi}{2}, \quad \int_0^\infty \frac{\sin^3 x\, dx}{x} = \frac{\pi}{4},$$

$$\int_0^\infty \frac{\sin^5 x\, dx}{x} = \frac{3\pi}{16}, \quad \int_0^\infty \frac{\sin^7 x\, dx}{x} = \frac{5\pi}{32}.$$

$$\therefore \int_0^\infty \tan^{-1}(m \sin x)\frac{dx}{x} = \frac{\pi}{2}(m - \tfrac{1}{6}m^3 + \tfrac{3}{40}m^5 - \tfrac{5}{112}m^7 + \ldots)$$

$$= \frac{\pi}{2}\log[m + \sqrt{(1+m^2)}].$$

$$\therefore \int_0^\infty \tan^{-1}(\tan\alpha \sin x)\frac{dx}{x} = \frac{\pi}{2}\log(\tan\alpha + \sec\alpha).$$

II. Solution by J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md.

According to Frullani's Theorem (see Williamson's *Integral Calculus*, 119, page 155), we have

$$\int_0^\infty \frac{\tan^{-1}ax - \tan^{-1}bx}{x}dx = \frac{\pi}{2}\log\frac{a}{b}.$$

Putting in the proposed integral $\tan\alpha = m$, we have

$$\int_0^\infty \tan^{-1}(m \sin x)\frac{dx}{x}$$

$$= \int_0^\infty \{\tan^{-1}[\sqrt{(1+m^2)} + m]\sin x - \tan^{-1}[\sqrt{(1+m^2)} - m]\sin\}\frac{dx}{x}$$

$$= \frac{\pi}{2}\log\frac{\sqrt{(1+m^2)} + m}{\sqrt{(1+m^2)} - m}.$$

Substituting $\tan\alpha$ for m, we get

$$\frac{\pi}{2}\log\frac{1+\sin\alpha}{1-\sin\alpha}=\frac{\pi}{2}\log\frac{(1+\sin\alpha)^2}{1-\sin^2\alpha}=\frac{\pi}{2}\log\left(\frac{1+\sin\alpha}{\cos\alpha}\right)^2=\pi\log(\tan\alpha+\sec\alpha),$$

which result is quite as great as that given in the proposed problem.

MECHANICS.

188. Proposed by H. L. ORCHARD, M. A., B. Sc.

Spherical bubbles are rising in water. Find the relation between radius and velocity.

No solution has been received.

189 (Incorrectly numbered 190). Proposed by DR. L. E. DICKSON, The University of Chicago, Chicago, Ill.

Give the axiomatic principle of Physics which is equivalent to the theorem on the compound of two circles ("Graphical Methods in Trigonometry," MONTHLY, June-July, 1905).

No solution has been received.

190 (Incorrectly numbered 191). Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

A pole hinged at the bottom leans against the mid-point of a smooth rope suspended from two supports of equal height. Determine the position of equilibrium.

Solution by G. W. GREENWOOD, M. A., Dunbar, Pa.

Assuming that the figure is symmetrical with respect to the plane bisecting perpendicularly the segment determined by the two points of support, the tensions of the portions of the rope resolved parallel to the pole are equal in magnitude and direction, and must therefore be zero since the rope is smooth. Hence, the pole rests in a position perpendicular to the plane determined by the portions of the rope.

191 (Incorrectly numbered 192). Proposed by REV. J. H. MEYER, S. J., College of Sacred Heart, Augusta, Ga.

Find the velocity of a planet at a given point in its orbit.

Solution by G. W. GREENWOOD, M. A., Dunbar, Pa.

In the case of any central orbit we have $v=h/p\ldots(1)$, where p is the perpendicular from the pole upon the tangent at the point, and h is a constant. Also,

$$\frac{1}{p^2}=u^2+\left(\frac{du}{d\theta}\right)^2$$

and the equation of an ellipse, referred to a focus, is $lu=1+e\cos\theta$, where l is the semi-latus rectum.

$$\therefore \frac{1}{p^2}=\frac{2(1+e\cos\theta)-(1-e^2)}{l^2}=\frac{2u}{l}-\frac{1}{al}.$$

Hence (1) may be written $v^2=\frac{h^2}{l}\left(\frac{2}{r}-\frac{1}{a}\right)$.

Also solved by the Proposer.

DIOPHANTINE ANALYSIS.

132. Proposed by DR. OSWALD WEBLEN, Princeton University, Princeton, N. J.

From the numbers, 0, 1, 2, ..., 42, select seven, such that the 42 differences of these seven numbers shall be congruent (mod. 43) to the numbers 0, 1, 2, ..., 42. The differences may be both positive and negative.

Discussion by F. H. SAFFORD, Ph. D., University of Pennsylvania.

For convenience, let a circle be divided at 43 points, numbered consecutively from 0 to 42. Take any seven of these *numbers* and call the number of spaces between adjacent numbers *intervals*. The 42 differences of these seven numbers are obtained by taking the seven intervals together with all possible sums of *adjacent* intervals, two, three, ..., six, at a time. All differences may be considered positive by this method.

In order that the seven *numbers* may be a solution it is necessary and sufficient that the *intervals* shall possess the following properties:

(1) They shall be so selected that their sum is 43.

(2) They shall be capable of being arranged so that their sums, taken one at a time, two adjacent, three adjacent, etc., shall all be different. It will be evident that no interval can be 0, otherwise the differences will not be all distinct. Also 1 and 2 must be intervals, while if 3 is not an interval 4 must be, in order to provide for the differences 1, 2, 3, 4.

It is not difficult to write the totality of selections for (1), and there are 77 of them.

In permuting each of the 77 sets, it will be found by trial that no set will satisfy (2). Hence the problem is impossible.*

AVERAGE AND PROBABILITY.

160. Proposed by J. F. LAWRENCE, A.B., Professor of Mathematics, Oklahoma Agricultural College, Stillwater, Okla.

Two points are taken at random in a triangle, the line joining them dividing the triangle into two portions. Find the mean value of that portion containing the center of gravity.

No solution has been received.

*There are solutions of the corresponding problem in which 43 is replaced by 3, 7, 13, or 21. See proposed problem 142. Ed. D.

163. Proposed by R. D. CARMICHAEL, Anniston, Ala.

In a regular n-gon a triangle is formed by taking three vertices at random. What is the mean value of the triangle?

No solution has been received.

164. Proposed by J. O. MAHONEY, B. E., M. Sc., Central High School, Dallas, Tex.

If m is prime, and the numbers 0, 1, 2, ..., m^2-1 are placed at random in the form of a square, the probability that the square is hyper-magic is $(m-1)m/(m^2-2)!$

No solution has been received.

167. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

A line l is divided into n segments by $n-1$ points taken at random on it; find the mean value of the product of p of the segments, the p segments being taken at random and p being less than n.

Solution by the PROPOSER.

Let a, b, c, ... be the p segments in some particular case. The chance that another point taken at random shall fall on the first of these segments is a/l; on the second, b/l; and so on. Hence the chance in this particular case that p new points taken at random will fall each on one of the p segments a, b, c, ... in an assigned order is $abc.../l^p$. The chance that they shall so fall in any order is evidently $p!abc.../l^p$. Hence the probability of this occurring, however the line is divided and however the p points are chosen, is $p!m(abc...)/l^p$.

If now we can find another expression involving no unknown quantity and giving this same probability, the two will enable us to determine the value of $m(abc...)$.

The number of ways in which p points can be taken one on each of p chosen segments is the same as the number of ways in which $p-1$ points can be placed one between the two of each consecutive pair of p other points, and is easily found by the theory of permutations to be $p!(p-1)!$. But p segments may be chosen from n segments in $\dfrac{n!}{p!(n-p)!}$ ways. Hence the whole number of ways in which p points can be chosen on n segments one on each of p segments is

$$\frac{p!(p-1)!\,n!}{p!(n-p)!}=\frac{n!(p-1)!}{(n-p)!}.$$

Now the whole number of ways in which $n-1+p$ points can be arranged is $(n+p-1)!$. Hence the chance that no two of the p points last chosen shall be on the same one of the original n segments is

$$\frac{n!(p-1)!}{(n-p)!}\div(n+p-1)!=\frac{n!(p-1)!}{(n-p)!(n+p-1)!}.$$

But this is the same chance as that which was found above in different terms. Equating the two, we have

$$\frac{p!\, m(abc\ldots)}{l^p} = \frac{n!(p-1)!}{(n-p)!(n+p-1)!}.$$

Hence, $m(abc\ldots) = \dfrac{l^p . n!(p-1)!}{p!(n-p)!(n+p-1)!}$.

For the case $p=n$ this value reduces to $\dfrac{l^n(n-1)!}{(2n-1)!}$, the mean value of the product of the n segments of the line. This result agrees with that found by Mr. Crofton in the solution of the latter problem in the Encyclopedia Britannica, Vol. XIX, p. 784.

This solution should have appeared instead of the solution published last month. Both solutions were received before the last issue went to press. By an oversight the defective solution got in with the material for publication. Ed. F.

170. Proposed by LON C. WALKER, Santa Barbara, Cal.

Find the area of a triangle formed by drawing a line at random through each of three points taken at random within a given triangle.

No solution has been received.

174. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

A chord of length C is drawn at random in a given ellipse. What is the average area of the segment cut off by the chord?

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Refer the ellipse to conjugate diameters so that $x^2/m^2+y^2/n^2=1$ is its equation.

$$\text{Area of segment} = \frac{2n}{m}\sin\omega \int_{(m/2n)\sqrt{(4n^2-c^2)}}^{m} \sqrt{(m^2-x^2)}\,dx$$

$$= mn\sin\omega\left[\cos^{-1}\left(\frac{\sqrt{(4n^2-c^2)}}{2n}\right) - \frac{c\sqrt{(4n^2-c^2)}}{4n^2}\right]$$

$$= ab\left[\cos^{-1}\left(\frac{\sqrt{(4n^2-c^2)}}{2n}\right) - \frac{c\sqrt{(4n^2-c^2)}}{4n^2}\right] = A.$$

ω is the inclination of m, n.

$$\text{Average area is } \triangle = \frac{\int_b^a A\,dn}{\int_b^a dn}.$$

$$\therefore \triangle = \frac{ab}{a-b}\left[a\cos^{-1}\left(\frac{\sqrt{(4a^2-c^2)}}{2a}\right) - b\cos^{-1}\left(\frac{\sqrt{(4b^2-c^2)}}{2b}\right)\right.$$
$$\left. + \frac{c}{4}\left(\frac{\sqrt{(4a^2-c^2)}}{a} - \frac{\sqrt{(4b^2-c^2)}}{b}\right)\right].$$

Corollary. If $a=b$, $\triangle = a^2\left[\cos^{-1}\left(\frac{\sqrt{(4a^2-c^2)}}{2a}\right) - \frac{c\sqrt{(4a^2-c^2)}}{4a^2}\right]$.

PROBLEMS FOR SOLUTION.

ALGEBRA.

275. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Given the simultaneous equations $x^y - y^x = 0$ and $y - x = a(a+1)^{1/a}$; find a solution which is real when $a > -1$.

276. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

If $x_1, x_2, \ldots, x_n$ be unequal and $f(n)$ be a rational integral function of degree $\geqslant n-2$, then shall

$$\sum_{r=n-1}^{r=1} \frac{f(x_r)}{(x_r - x_1)(x_r - x_2)\ldots(x_r - x_n)} = 0.$$

277. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

If $\alpha, \beta, \gamma, \delta$ are the roots of the quartic $ax^4 + bx^3 + cx^2 + dx + e = 0$, calculate the value of the product of the twelve expressions of the form $(4\alpha - 2\beta - \gamma - \delta)$ in terms of H, I, J, the well known functions of the differences of the roots.

GEOMETRY.

306. Proposed by J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md.

Find the length of the perpendicular let fall from the point in space (5, 6, 7) upon the line $x = 2z - 3$, and $y = -3z + 1$.

307. Proposed by WALTER D. LAMBERT, 416 B Street N. E., Washington, D. C.

A family of planes containing a common line intersects a sphere. Find the orthogonal trajectories of the traces. An analytic solution is preferred.

308. Proposed by W. J. GREENSTREET, M. A., Editor of the Mathematical Gazette, Stroud, England.

Find the locus of O, if the differences of the squares of tangents from it to circles A, B, C are x^2, y^2, z^2, respectively.

CALCULUS.

232. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Evaluate (a) $\int_0^{\frac{1}{2}\pi} \frac{\sin nx}{\sin x}dx$, and (b) $\int_0^{\frac{1}{2}\pi} \frac{\sin^2 nx}{\sin x}dx$, where n is a positive integer.

233. Proposed by W. J. GREENSTREET, M. A., Editor of the Mathematical Gazette, Stroud, England.

Prove that $\int_0^{\infty} \frac{a^{-1}dx}{1+2x\cos\theta+x^2} = \frac{\pi \sin(1-a)\theta}{\sin a\pi \sin\theta}$.

MECHANICS.

196. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

From a uniform, solid right circular cone two planes cut off a portion such that the sections are similar ellipses with co-planar axes (not parallel). The centers of the elliptic faces are O_1, O_2, and the center of gravity of the solid is G. GX parallel to O_1O_2 cuts the axis of the cone in X. Find GX/O_1O_2 in terms of the ratio of the major axes of the ellipses.

197. Proposed by WALTER D. LAMBERT, 416 B Street N. E., Washington, D. C.

Suppose that a primary planet and its satellite revolve with uniform angular velocity in circular orbits in the same plane. What relation must hold between the radii of their orbits and their angular velocities in order that the curve traced by the satellite shall be everywhere concave to the sun? Apply to the earth-moon system to prove that the moon's path is always concave to the sun.

DIOPHANTINE ANALYSIS.

142. Proposed by DR. L. E. DICKSON, The University of Chicago.*

Let n be an integer >1 and set $p=n(n-1)+1$. Required n integers whose $n(n-1)$ differences are congruent (modulo p) to the numbers $1, 2, \ldots, p-1$. Exhibit at least for $n=3, 4, 5$, all inequivalent sets of solutions where a set $a_1, a_2, \ldots, a_n$ is called equivalent to the set $m(a_1-d), m(a_2-d), \ldots, m(a_n-d)$, for any integers m and d (m not divisible by p).

143. Proposed by JOHN D. WILLIAMS (being the first of his 14 challenge problems proposed in 1832).

Make $x^2+y^2=a^2=z^2+w^2$ and $x^2-w^2=z^2-y^2=\square$.

144. Proposed by JOHN D. WILLIAMS (being the ninth of his 14 challenge problems proposed in 1832).

Make $(m^2+n^2)^2x^2 \pm (m^2+n^2)x=\square$, $(m^2-n^2)^2x^2 \pm (m^2-n^2)x=\square$, and $4m^2n^2x^2 \pm 2mnx=\square$.

*See problem 132, Diophantine Analysis, proposed by Dr. Veblen, and its discussion on page 215 by Dr. Safford.

AVERAGE AND PROBABILITY.

183. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

A point within a given triangle is joined to each of the corners. What is the average of the sum of the lengths of these three lines?

184. Proposed by HENRY HEATON, Atlantic, Iowa.

Through every point of the sides of a given square, straight lines are drawn across the square in every possible direction. What is their average length?

BOOKS AND PERIODICALS.

A Short Course in Differential Equations. By Donald Francis Campbell, Ph. D., Professor of Mathematics, Armour Institute of Technology. 8vo. Cloth, vii+96 pages. New York: The Macmillán Co.

The treatment of the subject of Differential Equations in this little book is sufficiently comprehensive to give the student of Engineering a working knowledge of the subject and to enable him to handle intelligently nearly all differential equations which he is likely to encounter. At the end of each chapter is a list of well chosen problems, many of them drawn from practical affairs in Electrical and Mechanical Engineering. The book is one that will meet the wants of that class of students who have not the time nor the desire to master the more comprehensive treatises of Johnson, Boole, Forsythe, etc. B.F.F.

An Introduction to Astronomy. By Forest Ray Moulton, Ph. D., Assistant Professor of Mathematics in The University of Chicago; and Author of An Introduction to Celestial Mechanics. 8vo. Half Leather, xviii+557 pages. New York: The Macmillan Co.

In this volume, the author aims to give an introductory account of the present state of the Science of Astronomy, to present the subject in such a way that it shall be easily comprehended by students who have only a knowledge of elementary algebra and geometry and who have no extensive scientific training, and to give them a well balanced conception of the Astronomy of the present day. The author treats the subject with the same enthusiasm that characterizes his teaching, and he has told the story of Astronomy in a most interesting and fascinating way. One reads with profound interest the chapter on Evolution of the Solar System, in which is given the various inconsistencies of the Laplacian, or Nebular, Hypothesis, and in which the author advances what he calls the "Spiral Nebular Hypothesis," a theory which was first advanced by Professor Chamberlain and the author. This interesting theory explains many facts that the Laplacian Theory does not explain. The theory is supported by many facts and is being recognized by many of the leading astronomers. This book in the hands of the student with a competent instructor to lead him can not fail to arouse great interest in that most wonderful of all sciences, astronomy. B. F. F.

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ON A CERTAIN CLASS OF CURVES GIVEN BY TRANSCENDENTAL EQUATIONS.

By R. D. CARMICHAEL, Professor of Mathematics, Presbyterian College, Anniston, Alabama.

1. DEFINITION. The equation

$$y^{1/y}=(fx)^{1/fx} \tag{1}$$

is satisfied both by $y=fx$ and by further sets of values of which two examples are $y=4$ or 2, $fx=2$ or 4. Now, $y=fx$ is defined as the *parent* curve, and the other locus is called the *companion* curve.

In a similar manner we define the two curves given by

$$(fy)^{1/fy}=(fx)^{1/fx}, \text{ and also by } [f(x, y)]^{1/f(x, y)}=[F(x, y)]^{1/F(x, y)}.$$

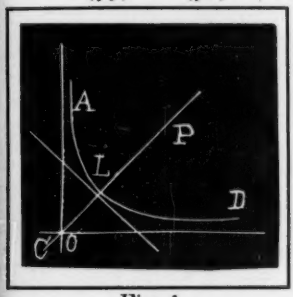

Fig. 1.

EXAMPLES. Figure 1 gives the locus of $y^{1/y}=x^{1/x}$. PC is the straight line $y=x$, and ALD is its companion curve. It is symmetrical with reference to PC. Its asymptotes are $x=1$ and $y=1$.

The locus of $(y^2)^{1/y^2}=(x^2)^{1/x^2}$ has a branch in each quadrant resembling in form ALD above. The figure possesses fourfold symmetry.

As a variation, we plot (Figure 2) the polar companion to the circle, giving the locus of $r^{1/r}=(10\cos\theta)^{1/10\cos\theta}$. The perpendicular to OQ at O is the asymptote in both directions.